THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2024-2025
Assignment 2
Due Date: October 14, 2024

1. Solve the following PDE using the spectral method:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), \quad (x,t) \in [0,1] \times (-\infty,\infty) \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = 100, \text{ for } x \in (0,1) \end{cases}$$

   (Hint: Odd extension may help.)

2. A discrete complex-valued function f can be represented by a vector $(f_0, f_1, \ldots, f_{n-1})^T$. Consider a matrix M where the entry in the j-th row and k-th column is given by $M_{jk} = e^{i\frac{2jk\pi}{n}}$.

   Please express the function f as a linear combination of the column vectors of M. In other words, you need to determine the coefficients for this linear combination.

3. Let $f(x)$ be a 2π-periodic complex-valued function and $\int_0^{2\pi} |f(x)|^2 dx < \infty$. Its complex Fourier coefficient is computed by $\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$ and complex Fourier series is

$$\mathcal{F}(f)(x) := \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx}$$

   and the truncated version is

$$\mathcal{F}_N(f)(x) := \sum_{k=-N}^{N} \hat{f}_k e^{ikx}$$

   Recall its real Fourier series is $a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$.Prove that

   (a) $\hat{f}_k = \frac{a_k - ib_k}{2}$, if $k \geq 1$

   (b) $\hat{f}_k = \frac{a_{-k} + ib_{-k}}{2}$, if $k \leq -1$

   (c) If $f(x)$ is real-valued, $a_k = 2\mathbf{Re}(\hat{f}_k)$ and $b_k = -2\mathbf{Im}(\hat{f}_k)$ for $k \geq 1$

4. Given a positive even integer N, let $E_k(x) = e^{ikx}$ for $k \geq 0$ and $x_j = j\frac{2\pi}{N}$ for $0 \leq j \leq N-1$. Since $E_k(x_j) = E_{k+N}(x_j)$, we can do discrete Fourier transform with the set of functions $\left\{E_k(x)\colon k = -\frac{N}{2}+1, -\frac{N}{2}+2, ..., \frac{N}{2}\right\}$.

For symmetry, we would like to do discrete Fourier transform with $E = \left\{E_k(x) \colon k = -\frac{N}{2}, -\frac{N}{2}+1, ..., \frac{N}{2}-1, \frac{N}{2}\right\}$ and updated computing rule is

$$\hat{f}_k = \frac{1}{a_k} \cdot \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) e^{-ikx_j}, \text{ for } k = -\tfrac{N}{2}, ..., \tfrac{N}{2}$$

where $a_k = 2$ if $k = \pm\frac{N}{2}$ otherwise 1. And Its inverse Discrete Fourier transform is given by

$$\left(I_N(f)\right)(x) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{f}_k e^{ikx}$$

(a) Let $\tilde{f}_k = \frac{1}{N} \sum\limits_{j=0}^{N-1} f(x_j) e^{-ikx_j}$, for $k = 0, ..., N-1$. Prove that

  i. $\hat{f}_k = \tilde{f}_{k+N}$, for $k = -\frac{N}{2}+1, ..., -1$
  ii. $\hat{f}_{\pm\frac{N}{2}} = \frac{1}{2}\tilde{f}_{\frac{N}{2}}$
  iii. $\sum_{k=0}^{N-1} |\tilde{f}_k|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |f_k|^2$

(b) (Optional) Prove that

$$\left(I_N(f)\right)(x) = \sum_{j=0}^{N-1} f(x_j) g_j(x) \tag{1}$$

where

$$g_j(x) = \frac{1}{N} \sin\left(N\frac{x-x_j}{2}\right) \cot\left(\frac{x-x_j}{2}\right)$$

and $g_j(x_k) = 1$ if $j = k$ otherwise 0.

(c) (Optional) By using the nodal basis representation (1), we can compute the derivative of $f(x)$ by $f^{(m)}(x) \approx \left(I_N(f)\right)^{(m)}(x)$. Prove that Let $\mathbf{f}_N = \left(f(x_0), ..., f(x_{N-1})\right)^T$ and $\mathbf{f}_N^{(m)} = \left(f^{(m)}(x_0), ..., f^{(m)}(x_{N-1})\right)^T$, then $\mathbf{f}_N^{(m)} = D^m \mathbf{f}_N$ for some matrix D^m. In particular,

$$D^1(k,j) = g_j'(x_k) = \begin{cases} \frac{(-1)^{k+j}}{2} \cot\left(\frac{(k-j)\pi}{N}\right), & \text{if } \quad k \neq j \\ 0, & \text{if } \quad k = j \end{cases}$$

5. Consider the differential equation:

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} = f(x) \text{ for } x \in (0, 2\pi),$$

where $a, b > 0$. Assume u and f are periodically extended to R. Divide the interval $[0, 2\pi]$ into n equal portions and let $x_j = \frac{2\pi j}{n}$ for $j = 0, 1, 2, ..., n-1$.

Let $\mathbf{u} = (u(x_0), u(x_1), ..., u(x_{n-1}))^T$ and $\mathbf{f} = (f(x_0), f(x_1), ..., f(x_{n-1}))^T$. for $j = 0, 1, 2, ..., n-1$.

(a) Use $u(x_{j\pm2)}$ to approximate $u'(x_j)$ and use $u(x_{j\pm4})$ and $u(x_j)$ to approximate $u''(x_j)$ and explain why the corresponding matrices $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively.

(b) Prove that $\overrightarrow{e^{ikx}} := (e^{ikx_0}, e^{ikx_1}, ..., e^{ikx_{n-1}})^T$ is an eigenvector of both $\mathcal{D}_1$ and $\mathcal{D}_2$ for $k = 0, 1, 2, ..., n-1$. What are their corresponding eigenvalues? Please explain your answer with details.

(c) Show that $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ forms a basis for C^n.

(d) Let $\mathbf{u} = \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\mathbf{f} = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$, where $\hat{u}_k, \hat{f}_k \in C$. If $\mathbf{u}$ satisfies $a\mathcal{D}_2\mathbf{u} + b\mathcal{D}_1\mathbf{u} = \mathbf{f}$, show that

$$(a\lambda_k^2 + b\lambda_k)\hat{u}_k = \hat{f}_k \text{ where } \lambda_k = i\frac{\sin(2kh)}{2h},$$

for $k = 0, 1, 2, ..., n-1$. Please explain your answer with details.